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Discrete improper affine spheres

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Abstract

The purpose of this paper is to discretize improper affine spheres and to investigate them in detail. We clarify a link between discrete improper affine spheres and Hirota's discrete Liouville equation, and characterize those surfaces in terms of loop groups.

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1. Introduction

In recent years, there has been explosive progress in the theory of discrete integrable systems. In this connection, discrete surfaces have been studied one after another with strong ties to physics and great potential for computer analysis. Those relationship between geometry and integrable systems can be diagrammed as below: the right arrows mean integrability conditions, the left ones geometric correspondent, the up ones continuum limit, and the down ones discretization. In general, one differential equation may have many discrete models, so how can we find a good one among them? A possible strategy is to discretize it via geometry. Such a link between discrete integrable systems and particular classes of discrete surfaces has been established. For example, Hirota's discrete sine-Gordon equation arises as the discrete integrability condition for discrete pseudo-spherical surfaces [1,7].

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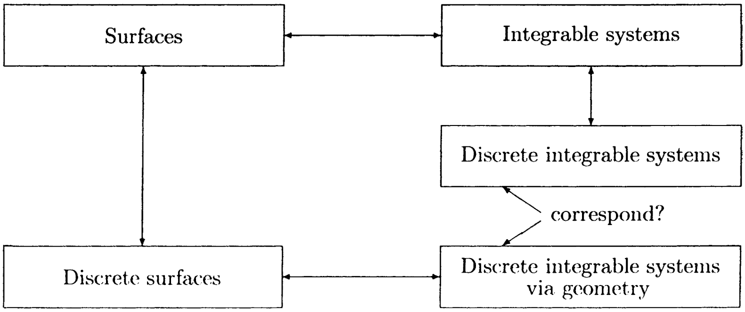

Affine spheres are those surfaces for which the affine shape operator is a scalar multiple of the indentity, but unlike the Euclidean case they are by no means simple or easy to determine [6]. A discrete integrable analogue of *proper* affine spheres was given by Bobenko and Schief [2,3], who presented a natural geometric discretization of them and investigated the corresponding discrete Gauss–Codazzi equations in detail. But, improper affine spheres make also an abundant and important class including ruled surfaces. Furthermore, every solution to the Liouville equation

$$(\log \omega)_{uv} + \omega^{-2} = 0 \tag{1}$$

describes an improper affine sphere in $\mathbb{R}^3$. The solution ω of (1) becomes the volume element of the affine metric. In this paper, we discretize *improper* affine spheres. A discrete integrable analogue of the Eq. (1) was constructed by Hirota [5] without using any relation to geometry. We show that Hirota's discrete Liouville equation

$$2\sinh \frac{W_{12} - W_1 - W_2 + W}{2} + \exp \frac{-W_{12} - W_1 - W_2 - W}{2} = 0 \tag{2}$$

describes our discrete improper affine spheres, and this observation permits the diagram given on the previous page to commute. Moreover, in Section 6, we characterize improper affine spheres with indefinite affine metric in terms of loop groups, and give a discrete analogue of that characterization, which shows that our discrete improper affine spheres also have rich algebraic structure.

2. Preliminary

In this section, let us recall basic notation of affine differential geometry. Let M be a two-dimensional smooth manifold and D the usual flat affine connection on $\mathbb{R}^3$. For an immersion $f : M \to (\mathbb{R}^3, D)$, we choose an arbitrary transversal vector field ξ on M, that is

$$T_{f(x)}\mathbb{R}^3 = f_*(T_x M) \oplus \mathbb{R}\xi_x$$

at each point $x \in M$. The formulas of Gauss and Weingarten

$$D_X(f_*Y) = f_*(\nabla_X Y) + h(X, Y)\xi, \qquad D_X\xi = -f_*(SX) + \tau(X)\xi$$

induce on M an affine connection ∇, a symmetric $(0, 2)$-tensor field h, a $(1, 1)$-tensor field S and a 1-form τ. The determinant function of $\mathbb{R}^3$ induces a volume form θ on M via

$$\theta(X, Y) = \det(f_*X, f_*Y, \xi).$$

The rank of the affine fundamental form h is independent of the choice of transversal vector field ξ. We assume that the rank is 2, so that h can be treated as a nondegenerate metric on M. This is a basic assumption on which Blaschke developed affine differential geometry of hypersurfaces. For each point $x \in M$, there is a transversal field ξ defined in a neighbourhood of x satisfying the conditions

$$\omega = \theta, \qquad \nabla\theta = 0. \tag{3}$$

Here ω denotes the volume element of the nondegenerate metric h. The former is called volume condition and the latter equiaffine condition. Since the determinant function is parallel relative to D, the equation $\nabla\theta = \tau\theta$ holds. Therefore, the equiaffine condition is equivalent to $\tau = 0$.

A transversal field satisfying (3) is called a *Blaschke normal field*, which is uniquely determined up to sign locally. The immersion $f : (M, \nabla) \to (\mathbb{R}^3, D)$ with Blaschke normal field is called *Blaschke immersion* and h is called *affine metric*.

Lemma 2.1. *The Laplacian of a Blaschke immersion,* Δf *relative to the affine metric is equal to* 2ξ.

Definition 2.2. A Blaschke immersion f is called an *improper* affine sphere if S is identically 0. If $S = \lambda I$, where λ is a nonzero constant, then f is called a *proper* affine sphere.

An affine sphere has the following characteristic property (cf. [6, p. 43]), which helps us to discretize affine spheres.

Lemma 2.3. *Let* $f : M \to \mathbb{R}^3$ *be a Blaschke immersion. Then* (f, M) *is an improper affine sphere if and only if the Blaschke normals are parallel in* $\mathbb{R}^3$, *and* (f, M) *is a proper affine sphere if and only if the Blaschke normals meet at one point in* $\mathbb{R}^3$.

Since a discretization of surfaces essentially depends on a choice of a coordinate system, we need consider separately the cases of which metric is indefinite or definite.

3. Discrete indefinite improper affine sphere

Assume now that a Blaschke immersion f is an improper affine sphere and the affine metric h is indefinite. We shall also say f an indefinite improper affine sphere. We choose an asymptotic coordinate system $(\mathbb{D}, (u, v))$ with respect to h so that $h = 2\omega\, \mathrm{d}u\, \mathrm{d}v$. By the volume condition, we have that $\omega(u, v) = \det(f_u, f_v, \xi)$. Applying if necessary a transformation $(u, v) \mapsto (v, -u)$, we can always achieve $\omega > 0$.

Proposition 3.1. *Let $f : \mathbb{D} \subset M \to \mathbb{R}^3$ be an indefinite improper affine sphere. Then the Gauss equations are as follows:*

$$f_{uu} = \frac{\omega_u}{\omega} f_u + \frac{a}{\omega} f_v, \tag{4}$$

$$f_{uv} = \omega\xi, \tag{5}$$

$$f_{vv} = \frac{b}{\omega} f_u + \frac{\omega_v}{\omega} f_v, \tag{6}$$

where ξ is a nonzero constant vector in $\mathbb{R}^3$, and three functions a, b and ω satisfy the Gauss–Codazzi equations

$$(\log\omega)_{uv} + ab\omega^{-2} = 0, \quad a_v = 0, \quad b_u = 0. \tag{7}$$

Proof. We choose asymptotic coordinate system $(\mathbb{D}, (u, v))$ and obtain

$$f_{uv} = \left| h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)\right| \xi = \omega\xi$$

by Lemma 2.1. □

Since a is a function only in u, we obtain $a\,\mathrm{d}u^3 = \mathrm{d}\tilde{u}^3$, where $\tilde{u} = \int a^{1/3}\mathrm{d}u$. Namely, we can take $a = 1 = b$ without a loss of generality in the case that $ab \neq 0$. Then, the compatibility condition (7) is reduced to the Liouville equation

$$w_{uv} + \mathrm{e}^{-2w} = 0, \tag{8}$$

where $\omega = \mathrm{e}^w$. Consider the relations

$$(\tilde{w} - w)_u = -\beta\,\mathrm{e}^{-\tilde{w}-w}, \qquad (\tilde{w} + w)_v = \frac{1}{\beta}\mathrm{e}^{\tilde{w}-w}, \tag{9}$$

where $\beta \in \mathbb{R}$ is a nonzero constant which is known as a Bäcklund parameter. The integrability condition of (9) produces $\tilde{w}_{uv} = 0$. Thus, the implicit relations (9) give a link between the nonlinear equation and the linear equation. This connection may be exploited to solve the Liouville equation in full generality. Inserting the general solution $\tilde{w}(u, v) = p(u) + q(v)$ into the Bäcklund relations (9), and subsequent integration produces a general solution of the Liouville equation in the form

$$f(u, v) = \xi \int_{u_0}^{u}\int_{v_0}^{v}\left(\beta\int_{s_0}^{s} \mathrm{e}^{-2p(\sigma)}\mathrm{d}\sigma + \frac{1}{\beta}\int_{t_0}^{t}\mathrm{e}^{2q(\sigma)}\mathrm{d}\sigma + \alpha\right) \times \mathrm{e}^{p(s)-q(t)}\,\mathrm{d}t\,\mathrm{d}s + \eta(u) + \zeta(v),$$

where $\alpha \in \mathbb{R}$ is a constant and $\eta(u), \zeta(v)$ are vectors in $\mathbb{R}^3$.

In the case that $ab = 0$, we easily obtain

$$f(u, v) = \xi \int_{u_0}^{u} \mathrm{e}^{p(\sigma)}\,\mathrm{d}\sigma \int_{v_0}^{v} \mathrm{e}^{q(\sigma)}\,\mathrm{d}\sigma + \eta(u) + \zeta(v).$$

The following proposition is well known (cf. [6, pp. 92, 116]).

Proposition 3.2. *If f is a ruled improper affine sphere, then it is locally of the form $z = xy + \varphi(x)$, where φ is an arbitrary function of x. Conversely, the graph of $z = xy + \varphi(x)$ is a ruled improper affine sphere.*

Now we discretize improper affine spheres in a purely geometric manner. For a map $F : \mathbb{Z}^2 \to \mathbb{R}^3$, we denote increments of the discrete variables by subscripts, namely,

$$F = F(n,m), \qquad F_1 = F(n+1,m), \qquad F_2 = F(n,m+1),$$
$$F_{12} = F(n+1,m+1).$$

Moreover, decrements are indicated by subscripts with overbars, that is

$$F_{\bar{1}} = F(n-1,m), \qquad F_{\bar{2}} = F(n,m-1).$$

Taking the Gauss equations (4)–(6) into account, we give the following definition. Discretizing surfaces is nothing less than discretizing the coordinate system.

Definition 3.3. A map $F : \mathbb{Z}^2 \to \mathbb{R}^3$ is called a *discrete indefinite improper affine sphere* if it has the following properties: every five points $F = F(n,m)$ and its neighbours $F_1, F_2, F_{\bar{1}}, F_{\bar{2}}$ lie on one plane. The vectors $F_{12} + F - F_1 - F_2$ are all parallel in $\mathbb{R}^3$.

Proposition 3.4. *Let $F : \mathbb{Z}^2 \to \mathbb{R}^3$ be a discrete indefinite improper affine sphere. Then the discrete Gauss equations are as follows:*

$$(F_1 - F) - (F - F_{\bar{1}}) = \frac{\Omega - \Omega_{\bar{1}}}{\Omega}(F_1 - F) + \frac{A}{\Omega}(F_2 - F), \tag{10}$$

$$F_{12} + F - F_1 - F_2 = \Omega\, \Xi, \tag{11}$$

$$(F_2 - F) - (F - F_{\bar{2}}) = \frac{B}{\Omega}(F_1 - F) + \frac{\Omega - \Omega_{\bar{2}}}{\Omega}(F_2 - F), \tag{12}$$

where Ξ is a nonzero constant vector in $\mathbb{R}^3$ and three functions A, B and Ω satisfy the discrete Gauss–Codazzi equations

$$\Omega_{12}\Omega - \Omega_1\Omega_2 + A_1 B_2 = 0, \qquad A_2 - A = 0, \qquad B_1 - B = 0. \tag{13}$$

Moreover, these Eqs. (10)–(13) become continuous ones (4)–(7) in the continuum limit

$$F = f, \qquad \Omega = \omega\varepsilon_1\varepsilon_2, \qquad A = a\varepsilon_1^3, \qquad B = b\varepsilon_2^3, \tag{14}$$

where smooth variables are correlated to discrete ones as $(u, v) = (\varepsilon_1 n, \varepsilon_2 m)$ for small positive numbers ε_1 and ε_2.

Proof. Since F is a discrete indefinite improper affine sphere, there exist functions on $\mathbb{Z}^2$ such that

$$F_{11} - F_1 = P(F_1 - F) + Q(F_{12} - F_1), \qquad F_{12} + F - F_1 - F_2 = \Omega\,\Xi,$$
$$F_{22} - F_2 = R(F_2 - F) + S(F_{12} - F_2),$$

where Ξ is a nonzero constant vector in $\mathbb{R}^3$. The compatibility condition of F is equivalent to the system

$$0 = P_2 + Q_2 S - P, \qquad 0 = Q_2 R - Q,$$
$$0 = P_2 \Omega + Q_2 S \Omega + Q_2 \Omega_2 - Q\Omega - \Omega_1, \qquad 0 = S_1 P - S,$$
$$0 = R_1 + S_1 Q - R, \qquad 0 = R_1 \Omega + S_1 Q \Omega + S_1 \Omega_1 - S\Omega - \Omega_2.$$

The aimed equations (13) are obtained by setting $A_1 = Q\Omega$ and $B_2 = S\Omega$.

Next, we regard a discrete map F as an approximation of a smooth map f, that is

$$F(n, m) = f(\varepsilon_1 n, \varepsilon_2 m)$$

for small $\varepsilon_1, \varepsilon_2$, then, the Taylor expansions

$$F_1 - F = \varepsilon_1 f_u + \frac{\varepsilon_1^2}{2} f_{uu} + \mathrm{O}(\varepsilon_1^3), \qquad F_2 - F = \varepsilon_2 f_v + \frac{\varepsilon_2^2}{2} f_{vv} + \mathrm{O}(\varepsilon_2^3)$$

apply. Thus, the discrete Gauss and Gauss–Codazzi equations (10)–(13) produce continuous ones (4)–(7) in the continuum limit $\varepsilon_1, \varepsilon_2 \to 0$. □

In the case that $AB \neq 0$, the first equation of the systems (13) is locally written down as

$$2 \sinh \frac{W_{12} - W_1 - W_2 + W}{2} + A_1 B_2 \exp \frac{-W_{12} - W_1 - W_2 - W}{2} = 0, \tag{15}$$

where $\Omega = \pm \exp W$.

Remark 3.5. Eq. (15) is exactly Hirota's discrete Liouville equation (2) when $AB = 1$. He constructed a discrete integrable analogue to the Liouville equation (8) without using any relation to geometry, and it has been revealed that his method to discretize nonlinear partial differential equations produces good difference ones in the view of discrete integrability. Thus, our discrete indefinite improper affine spheres are described in terms of discrete integrable systems.

Remark 3.6. Hirota proposed a Bäcklund transformation for the discrete Liouville equation (2) as follows:

$$\sinh \frac{\tilde{W}_1 - \tilde{W} - W_1 + W}{2} + \beta \exp \frac{-\tilde{W}_1 - \tilde{W} - W_1 - W}{2} = 0, \tag{16}$$

$$\sinh \frac{\tilde{W}_2 - \tilde{W} + W_2 - W}{2} + \frac{1}{4\beta} \exp \frac{\tilde{W}_2 + \tilde{W} - W_2 - W}{2} = 0, \tag{17}$$

where β is a nonzero parameter and $\tilde{W}$ is a solution to the discrete wave equation

$$\tilde{W}_{12} - \tilde{W}_1 - \tilde{W}_2 + \tilde{W} = 0.$$

One can directly verify that if W is a solution to the systems (16) and (17), then W satisfies the discrete Liouville equation (2).

In the case that $AB = 0$, we obtain the following theorem, which is a discrete analogue of Proposition 3.2.

Theorem 3.7. *Let $F : \mathbb{Z}^2 \to \mathbb{R}^3$ be a ruled discrete indefinite improper affine sphere, that is the points $F(n, m_0)$ lie on a line for any fixed integer m_0. Then it is locally of the form*

$$(n, m) \mapsto {}^t(n, m, nm + \Phi(n)),$$

where Φ is an arbitrary sequence of n. Moreover it becomes the continuous graph $z = xy + \varphi(x)$ by taking an appropriate continuum limit.

Proof. We show that the difference systems (10)–(13) provide the theorem. From the Gauss equation (11), the vector $F(n, m)$ is of the form

$$F(n,m) = \begin{cases} \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} \Omega(i,j)\Xi + F(n,0) + F(0,m) - F(0,0), & n, m > 0, \\ -\sum_{i=n}^{-1}\sum_{j=0}^{m-1} \Omega(i,j)\Xi + F(n,0) + F(0,m) - F(0,0), & n < 0, m > 0, \\ F(n,m), & nm = 0, \\ \sum_{i=n}^{-1}\sum_{j=m}^{-1} \Omega(i,j)\Xi + F(n,0) + F(0,m) - F(0,0), & n, m < 0, \\ -\sum_{i=0}^{n-1}\sum_{j=m}^{-1} \Omega(i,j)\Xi + F(n,0) + F(0,m) - F(0,0), & n > 0, m < 0. \end{cases}$$

If $AB \neq 0$, a discrete indefinite improper affine sphere F cannot be ruled. Hence we can assume $B = 0$ without a loss of generality. The function $\Omega(n, m)$ is of the form

$$\Omega(0,0)\Omega(n,m) = P(n)Q(m),$$

where $P(n) = \Omega(n, 0)$ and $Q(m) = \Omega(0, m)$ are arbitrary one variable functions.

We can assume that the initial value $\Omega(0, 0)$ is equal to 1, and we set formally the summation $\sum_{k=k_2}^{k_1}$ to be always zero for $k_1 < k_2$. From the discrete Gauss equations (10) and (12), we have

$$F(0,m) = \begin{cases} \sum_{j=0}^{m-1} Q(j)(F(0,1) - F(0,0)) + F(0,0), & m > 0, \\ -\sum_{j=m}^{-1} Q(j)(F(0,1) - F(0,0)) + F(0,0), & m \le 0, \end{cases}$$

and

$$F(n,0) = \begin{cases} \sum_{l=0}^{n-1} P(l)(F(1,0) - F(0,0)) + \sum_{l=0}^{n-1} P(l) \\ \quad \times \sum_{k=0}^{l-1}\left(\dfrac{A(k+1)}{P(k)P(k+1)}\right)(F(0,1) - F(0,0)) + \sum_{l=0}^{n-1} P(l) \\ \quad \times \sum_{k=0}^{l-1}\left(\dfrac{A(k+1)}{P(k)P(k+1)}\right)\sum_{i=0}^{k} P(i)\Xi + F(0,0), & n > 0, \\ -\sum_{l=n}^{-1} P(l)\,(F(1,0) - F(0,0)) + \sum_{l=n}^{-1} P(l) \\ \quad \times \sum_{k=l}^{-1}\left(\dfrac{A(k+1)}{P(k)P(k+1)}\right)(F(0,1) - F(0,0)) - \sum_{l=n}^{-1} P(l) \\ \quad \times \sum_{k=l}^{-1}\left(\dfrac{A(k+1)}{P(k)P(k+1)}\right)\sum_{i=k+1}^{-1} P(i)\Xi + F(0,0), & n \le 0. \end{cases}$$

Then we obtain the following expressions: in the case that $n \geq 0$ and $m \geq 0$,

$$
\begin{aligned}
F(n,m) = {} & \sum_{i=0}^{n-1} P(i)(F(1,0)-F(0,0)) \\
& + \left(\sum_{j=0}^{m-1} Q(j) + \sum_{i=0}^{n-1} P(i) \sum_{k=0}^{i-1} \frac{A(k+1)}{P(k+1)P(k)} \right) (F(0,1)-F(0,0)) \\
& + \left(\sum_{i=0}^{n-1} P(i) \sum_{j=0}^{m-1} Q(j) + \sum_{i=0}^{n-1} P(i) \sum_{k=0}^{i-1} \frac{A(k+1)}{P(k+1)P(k)} \sum_{l=0}^{k} P(l) \right) \Xi + F(0,0).
\end{aligned}
$$

In the case that $n \leq 0$ and $m \geq 0$,

$$
\begin{aligned}
F(n,m) = {} & -\sum_{i=n}^{-1} P(i)(F(1,0)-F(0,0)) \\
& + \left(\sum_{j=0}^{m-1} Q(j) + \sum_{i=n}^{-1} P(i) \sum_{k=i}^{-1} \frac{A(k+1)}{P(k+1)P(k)} \right) (F(0,1)-F(0,0)) \\
& - \left(\sum_{i=n}^{-1} P(i) \sum_{j=0}^{m-1} Q(j) + \sum_{i=n}^{-1} P(i) \sum_{k=i}^{-1} \frac{A(k+1)}{P(k+1)P(k)} \sum_{l=k+1}^{-1} P(l) \right) \Xi + F(0,0).
\end{aligned}
$$

In the case that $n \leq 0$ and $m \leq 0$,

$$
\begin{aligned}
F(n,m) = {} & -\sum_{i=n}^{-1} P(i)(F(1,0)-F(0,0)) \\
& - \left(\sum_{j=m}^{-1} Q(j) - \sum_{i=n}^{-1} P(i) \sum_{k=i}^{-1} \frac{A(k+1)}{P(k+1)P(k)} \right) (F(0,1)-F(0,0)) \\
& + \left(\sum_{i=n}^{-1} P(i) \sum_{j=m}^{-1} Q(j) - \sum_{i=n}^{-1} P(i) \sum_{k=i}^{-1} \frac{A(k+1)}{P(k+1)P(k)} \sum_{l=k+1}^{-1} P(l) \right) \Xi + F(0,0).
\end{aligned}
$$

In the case that $n \geq 0$ and $m \leq 0$,

$$
\begin{aligned}
F(n,m) = {} & \sum_{i=0}^{n-1} P(i)(F(1,0) - F(0,0)) \\
& - \left(\sum_{j=m}^{-1} Q(j) - \sum_{i=0}^{n-1} P(i) \sum_{k=0}^{i-1} \frac{A(k+1)}{P(k+1)P(k)} \right) (F(0,1) - F(0,0)) \\
& - \left(\sum_{i=0}^{n-1} P(i) \sum_{j=m}^{-1} Q(j) - \sum_{i=0}^{n-1} P(i) \sum_{k=0}^{i-1} \frac{A(k+1)}{P(k+1)P(k)} \sum_{l=0}^{k} P(l) \right) \Xi + F(0,0).
\end{aligned}
$$

Thus a ruled discrete indefinite improper affine sphere is locally the graph $(n,m) \mapsto {}^t(n, m, nm + \Phi(n))$, where Φ is an arbitrary sequence of n.

Moreover, by regarding the functions P, Q and A as approximations of smooth functions p, q and a, respectively, via

$$
P(n) = \exp\left(p \left(u_0 + n \frac{u - u_0}{k-1} \right) \right) \frac{u - u_0}{k-1},
$$

$$
Q(m) = \exp\left(q \left(v_0 + m \frac{v - v_0}{k-1} \right) \right) \frac{v - v_0}{k-1},
$$

$$
A(n) = a \left(u_0 + n \frac{u - u_0}{k-1} \right) \left(\frac{u - u_0}{k-1} \right)^3,
$$

we obtain

$$
\lim_{k \to \infty} \sum_{n=0}^{k-1} P(n) = \int_{u_0}^{u} e^{p(\sigma)} \, d\sigma, \qquad \lim_{k \to \infty} \sum_{n=0}^{k-1} Q(m) = \int_{v_0}^{v} e^{q(\sigma)} \, d\sigma,
$$

and

$$
\lim_{k \to \infty} \sum_{n=0}^{k-1} \frac{A(n+1)}{P(n+1)P(n)} = \int_{u_0}^{u} d\sigma.
$$

Thus, F becomes the smooth graph $z = xy + \varphi(x)$ as k tends to infinity. □

4. Discrete definite improper affine sphere

Assume now that a Blaschke immersion f is an improper affine sphere and the affine metric h is definite. We shall also briefly say f a definite improper affine sphere. We choose an isothermal coordinate system $(\mathbb{D}, (u, v))$ with respect to h and set $z = x + iy$, so that $h = 2\omega \, dz \, d\bar{z}$. By the volume condition, we have that $\omega = -i \det(f_z, f_{\bar{z}}, \xi)$.

Proposition 4.1. *Let $f : \mathbb{D} \subset M \to \mathbb{R}^3$ be a definite improper affine sphere. Then the Gauss equations are as follows:*

$$f_{zz} = \frac{\omega_z}{\omega} f_z - \frac{a}{\omega} f_{\bar{z}}, \tag{18}$$

$$f_{z\bar{z}} = -\omega\xi, \tag{19}$$

$$f_{\bar{z}\bar{z}} = -\frac{\bar{a}}{\omega} f_z + \frac{\omega_{\bar{z}}}{\omega} f_{\bar{z}}, \tag{20}$$

where ξ is a nonzero constant vector in $\mathbb{R}^3$ and functions ω and a satisfy the Gauss–Codazzi equations

$$(\log \omega)_{z\bar{z}} + |a|^2\omega^{-2} = 0, \qquad a_{\bar{z}} = 0.$$

Taking the Gauss equations (18)–(20) into account, we give the following definition.

Definition 4.2. A map $F : \mathbb{Z}^2 \to \mathbb{R}^3$ is called a *discrete definite improper affine sphere* if it has the following properties: each quadrilateral F, F_1, F_{12}, F_2 is planar and the vector $F_1 + F_{\bar{1}} - F_2 - F_{\bar{2}}$ is tangential. The vectors $F_1 + F_{\bar{1}} + F_2 + F_{\bar{2}} - 4F$ are all parallel in $\mathbb{R}^3$.

Then we obtain the following theorem.

Theorem 4.3. *Let $F : \mathbb{Z}^2 \to \mathbb{R}^3$ be a discrete definite improper affine sphere. Then the discrete Gauss equations are as follows:*

$$(F_1 - F) - (F - F_{\bar{1}}) = A(F_1 - F) + B(F_2 - F) - \Omega\Xi,$$
$$F_{12} + F - F_1 - F_2 = P(F_1 - F) + Q(F_2 - F),$$
$$(F_2 - F) - (F - F_{\bar{2}}) = -A(F_1 - F) - B(F_2 - F) - \Omega\Xi,$$

where Ξ is a nonzero constant vector in $\mathbb{R}^3$ and five functions A, B, P, Q and Ω on $\mathbb{Z}^2$ satisfy the discrete Gauss–Codazzi equations

$$\Omega\frac{P+1}{A-1} = \Omega_2\frac{2B_2+1}{A_2-B_2-1}, \tag{21}$$

$$\Omega\frac{Q+1}{B+1} = \Omega_1\frac{2A_1-1}{A_1-B_1-1}, \tag{22}$$

$$\frac{P+1}{Q+1} = \frac{A_{12}-1}{2A_{12}-1}\frac{2B_{12}+1}{B_{12}+1}. \tag{23}$$

Proof. Since F is a discrete definite improper affine sphere, there exist five functions A, B, P, Q and Ω on $\mathbb{Z}^2$ such that

$$F_{12} + F - F_1 - F_2 = P(F_1 - F) + Q(F_2 - F),$$
$$F_1 + F_{\bar{1}} - F_2 - F_{\bar{2}} = 2A(F_1 - F) + 2B(F_2 - F),$$
$$F_1 + F_{\bar{1}} + F_2 + F_{\bar{2}} - 4F = -2\Omega\Xi,$$

where Ξ is a nonzero constant vector in $\mathbb{R}^3$. The discrete Gauss–Codazzi equations are:

$$\begin{pmatrix} \dfrac{P_1+1}{B_{12}+1}A_{12} \\ -\dfrac{\Omega_{12}}{B_{12}+1} \\ \dfrac{A_{12}Q_1-1}{B_{12}+1} \end{pmatrix} = \begin{pmatrix} -A_1+1 & \dfrac{A_1-1}{Q+1}P & 0 \\ -\Omega_1 & \dfrac{\Omega_1}{Q+1}P & 1 \\ -B_1 & \dfrac{B_1P+1}{Q+1} & 0 \end{pmatrix} \times \begin{pmatrix} -P_2(P+1)+(Q_2+1)\dfrac{P+1}{B_2+1}A_2 \\ -P_2Q+(Q_2+1)\dfrac{A_2Q-1}{B_2+1} \\ -(Q_2+1)\dfrac{\Omega_2}{B_2+1} \end{pmatrix},$$

$$\begin{pmatrix} \dfrac{Q_2+1}{A_{12}-1}B_{12} \\ \dfrac{\Omega_{12}}{A_{12}-1} \\ \dfrac{B_{12}P_2+1}{A_{12}-1} \end{pmatrix} = \begin{pmatrix} B_2+1 & -\dfrac{B_2+1}{P+1}Q & 0 \\ -\Omega_2 & \dfrac{\Omega_2}{P+1}Q & 1 \\ A_2 & -\dfrac{A_2Q-1}{P+1} & 0 \end{pmatrix} \times \begin{pmatrix} -(Q+1)Q_1+(P_1+1)\dfrac{Q+1}{A_1-1}B_1 \\ -PQ_1+(P_1+1)\dfrac{B_1P+1}{A_1-1} \\ (P_1+1)\dfrac{\Omega_1}{A_1-1} \end{pmatrix}.$$

Deleting P_1+1 and Q_2+1 from the systems above, we obtain the first two equations (21) and (22). Deleting P_2 and Q_1, we obtain that

$$\frac{Q+1}{A_1-1}\left(\frac{P+1}{B_2+1}\right)_1 = \left(\frac{Q+1}{A_1-1}\right)_2 \frac{P+1}{B_2+1}.$$

The combination of this equation with the systems (21) and (22) yields the last equation (23). □

5. Examples

We illustrate examples of discrete improper affine spheres.

Example 5.1 (Discrete hyperbolic paraboloid). The graph $z = (x^2 - y^2)/2$ is called *hyperbolic paraboloid*. The Gauss equations are $f_{xx} = \xi$, $f_{xy} = 0$, $f_{yy} = -\xi$, where $\xi = {}^t(0, 0, 1)$. Hence a hyperbolic paraboloid (Fig. 1) is an indefinite improper affine sphere. We

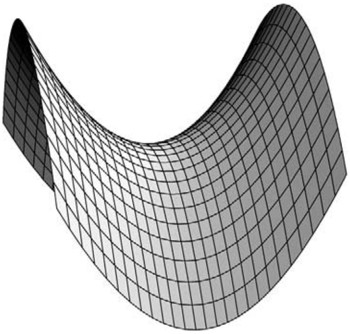

Fig. 1. Hyperbolic paraboloid.

choose an asymptotic coordinate system (u, v) and obtain $f(u, v) = {}^t(u + v, u - v, 2uv)$, where the Gauss equations are:

$$f_{uu} = 0, \qquad f_{uv} = 2\xi, \qquad f_{vv} = 0.$$

We call the map

$$F(n, m) = {}^t(n + m, n - m, 2nm)$$

discrete hyperbolic paraboloid (Fig. 2). The discrete Gauss equations are:

$$(F_1 - F) - (F - F_{\bar{1}}) = 0, \qquad F_{12} + F - F_1 - F_2 = 2\xi,$$
$$(F_2 - F) - (F - F_{\bar{2}}) = 0.$$

This is one of the simplest examples of discrete indefinite improper affine spheres.

Example 5.2 (Discrete Cayley surface). The graph $z = xy - x^3/3$ is called *Cayley surface*. There is a simple characterization of the Cayley surface, namely, if the cubic form $C = \nabla h$

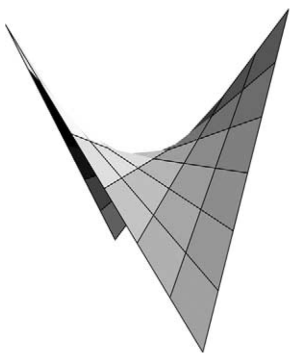

Fig. 2. Discrete hyperbolic paraboloid.

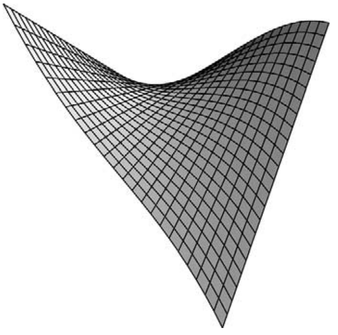

Fig. 3. Cayley surface.

is not 0 and parallel relative to ∇, a Blaschke immersion is affinely congruent to the Cayley surface (Fig. 3). We call the map

$$F(n,m) = {}^t\left(n, \frac{n^2-m^2}{2}, \frac{n^3-3nm^2}{6}\right)$$

discrete Cayley surface (Fig. 4). This is a ruled discrete indefinite improper affine sphere.

Example 5.3. The graph $z = xy + \cos x$ (Fig. 5), which is also a ruled improper affine sphere, has a discrete analogue as

$$F_\theta(n,m) = {}^t\left(n,\ \tan\frac{\theta}{2}\sin(\theta n), n\tan\frac{\theta}{2}\sin(\theta n) + \cos(\theta n)\right) \\ -\tan\frac{\theta}{2}\sin(\theta m)^t(0,1,n),$$

where θ is an arbitrary constant such that $\tan(\theta/2) \neq 0$. This is a ruled discrete indefinite improper affine sphere (Fig. 6).

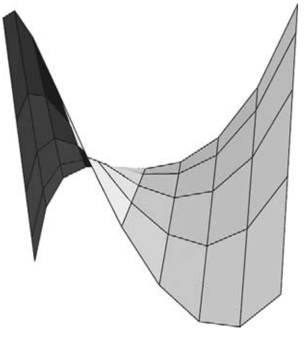

Fig. 4. Discrete Cayley surface.

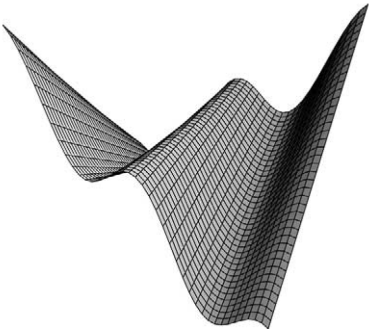

Fig. 5. The graph $z = xy + \cos x$.

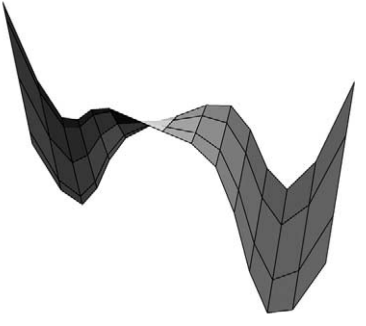

Fig. 6. Discrete surface $F_\theta(n, m)$.

Example 5.4 (Discrete elliptic paraboloid). The graph $z = (x^2 + y^2)/2$ is called *elliptic paraboloid* (Fig. 7). It is a definite improper affine sphere. The map

$$F(n, m) = {}^t\left(n, m, \frac{(n^2 + m^2)}{2}\right)$$

is called *discrete elliptic paraboloid* (Fig. 8.)

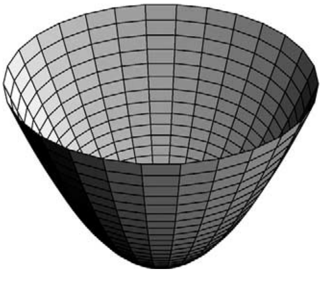

Fig. 7. Elliptic paraboloid.

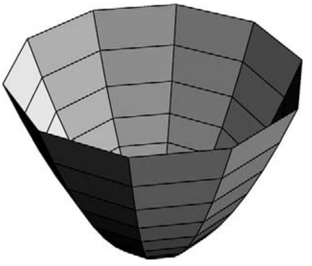

Fig. 8. Discrete elliptic paraboloid.

6. Loop group description

It is known that proper affine spheres can be described in terms of loop groups [3,4]. In this section, we show that improper affine spheres also allow such descriptions, and that a natural discretization of this description leads to the same definition of discrete improper affine spheres given in Section 3. Theorems 6.1 and 6.2 provide many smooth and discrete indefinite improper affine spheres systematically.

The Gauss equations (4)–(6) admit one parameter family of solutions, since the transformation $a \mapsto \mu a$ and $b \mapsto \mu^{-1} b$ with an arbitrary nonzero constant $\mu \in \mathbb{R}_*$ leave the Gauss–Codazzi equation (7) unchanged. Thus, we obtain moving frame equations where the coefficient matrices also depend on the additional real parameter μ. To make these coefficient matrices simple, we gauge the moving frame and replace μ by $\lambda = \sqrt[3]{\mu}$. We set

$$\varphi(u, v, \lambda) = (f_u, f_v, \xi) \begin{pmatrix} \dfrac{1}{\lambda\sqrt{\omega}} & 0 & 0 \\ 0 & \dfrac{\lambda}{\sqrt{\omega}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and obtain

$$\varphi(u, v, \lambda)^{-1} \varphi(u, v, \lambda)_u = \frac{\omega_u}{2\omega} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 & 0 \\ \dfrac{a}{\omega} & 0 & 0 \\ 0 & \sqrt{\omega} & 0 \end{pmatrix}, \tag{24}$$

$$\varphi(u, v, \lambda)^{-1} \varphi(u, v, \lambda)_v = \frac{\omega_v}{2\omega} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \lambda^{-1} \begin{pmatrix} 0 & \dfrac{b}{\omega} & 0 \\ 0 & 0 & 0 \\ \sqrt{\omega} & 0 & 0 \end{pmatrix}. \tag{25}$$

This map $\varphi : \mathbb{D} \times \mathbb{C}_* \to \mathrm{SL}\,(3, \mathbb{C})$ is called the *modified frame* of the indefinite improper affine sphere f. It turns out that a solution φ to the system above satisfies the relations

$$\varphi(u, v, \bar{\lambda}) = \overline{\varphi(u, v, \lambda)},$$
$$\varphi(u, v, q\lambda) = Q\varphi(u, v, \lambda) Q^{-1}, {}^t\varphi(u, v, -\lambda) T = T\varphi(u, v, \lambda)^{-1},$$

where

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} q & 0 & 0 \\ 0 & q^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad q = \mathrm{e}^{2\pi \mathrm{i}/3}.$$

Then we consider the map $\varphi : \mathbb{D} \times \mathbb{C}_* \to \mathrm{SL}\,(3, \mathbb{C})$ as $\varphi : \mathbb{D} \to \{\gamma : \mathbb{C}_* \to \mathrm{SL}\,(3, \mathbb{C})\}$, and introduce the group

$$G[\lambda] = \{\gamma : S^1 \to \mathrm{SL}\,(3, \mathbb{C}) | \gamma(\bar{\lambda}) = \overline{\gamma(\lambda)}, \gamma(q\lambda) = Q\gamma(\lambda)Q^{-1}, \\ {}^t\gamma(-\lambda)T = T\gamma(\lambda)^{-1}\}.$$

The second reduction should be understood in terms of a Fourier series expansion $\gamma(\lambda) = \sum_{k\in\mathbb{Z}} \lambda^k \gamma_k$ with coefficients of the form

$$\gamma_{3n} = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}, \quad \gamma_{3n+1} = \begin{pmatrix} 0 & 0 & * \\ * & 0 & 0 \\ 0 & * & 0 \end{pmatrix}, \quad \gamma_{3n+2} = \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & * \\ * & 0 & 0 \end{pmatrix}. \tag{26}$$

The Lie algebra of this group is

$$\mathfrak{g}[\lambda] = \{\xi : S^1 \to \mathfrak{sl}(3, \mathbb{C}) | \xi(\bar{\lambda}) = \overline{\xi(\lambda)}, \xi(q\lambda) = Q\xi(\lambda)Q^{-1}, {}^t\xi(-\lambda)T = -T\xi(\lambda)\}.$$

It is easily checked that $\varphi^{-1}\varphi_u$ and $\varphi^{-1}\varphi_v$ are in $\mathfrak{g}[\lambda]$, which implies that φ is in $G[\lambda]$ for all $(u, v) \in \mathbb{D}$. The following natural subgroups of $G[\lambda]$ are essential for the construction of discrete affine spheres.

$$G^0 = G^+[\lambda] \cap G^-[\lambda] = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SL}(3, \mathbb{C}) \middle| \alpha \in \mathbb{C}_* \right\},$$

where $G^+[\lambda] = \{\gamma \in G[\lambda] | \gamma(\lambda) = \sum_{k\geq 0} \lambda^k \gamma_k\}$ and $G^-[\lambda] = \{\gamma \in G[\lambda] | \gamma(\lambda) = \sum_{k\leq 0} \lambda^k \gamma_k\}$.

Then we obtain the following theorem which provides us a loop group description of indefinite improper affine spheres.

Theorem 6.1. *Let $\mathbb{D}$ be a domain in $\mathbb{R}^2$, and $\mathcal{U}$ and $\mathcal{V} : \mathbb{D} \to \mathfrak{g}[\lambda]$, two smooth maps of the form*

$$\mathcal{U}(\lambda) = \sum_{k=0}^{1} \lambda^k \mathcal{U}^{(k)}, \qquad \mathcal{V}(\lambda) = \sum_{k=0}^{1} \lambda^{-k} \mathcal{V}^{(k)},$$

with their coefficient functions $\mathcal{U}_{32}^{(1)}$ and $\mathcal{V}_{31}^{(1)}$ are positive on $\mathbb{D}$. Assume, moreover, for all loop parameters $\lambda \in S^1$,

$$[\mathcal{U}(\lambda), \mathcal{V}(\lambda)] - \mathcal{U}(\lambda)_v + \mathcal{V}(\lambda)_u = 0.$$

We define a map $\varphi : \mathbb{D} \to G[\lambda]$ *via the differential system*

$$\varphi(\lambda)^{-1}\varphi(\lambda)_u = \mathcal{U}(\lambda), \qquad \varphi(\lambda)^{-1}\varphi(\lambda)_v = \mathcal{V}(\lambda).$$

Then, there exists uniquely a matrix $\mathcal{C} \in G^0$ *such that* $\varphi\mathcal{C} : \mathbb{D} \to G[\lambda]$ *is a modified frame of indefinite improper affine spheres.*

Proof. We construct a modified frame $\psi(\lambda)$ which solves the system (24) and (25) for some three functions a, b and $\omega : \mathbb{D} \to \mathbb{R}$ satisfying Gauss–Codazzi equations (7). The map $\psi = \varphi\mathcal{C}$ satisfies

$$\psi(\lambda)^{-1}\psi(\lambda)_u = \mathcal{C}^{-1}\mathcal{U}^{(0)}\mathcal{C} + \mathcal{C}^{-1}\mathcal{C}_u + \lambda\mathcal{C}^{-1}\mathcal{U}^{(1)}\mathcal{C} = \hat{\mathcal{U}}^{(0)} + \lambda\hat{\mathcal{U}}^{(1)} = \hat{\mathcal{U}}(\lambda).$$

We define a 3×3 diagonal matrix $\mathcal{C} = \mathrm{diag}[\alpha, 1/\alpha, 1]$, where $\alpha = \sqrt{\mathcal{U}^{(1)}_{32}/\mathcal{V}^{(1)}_{31}}$, and we obtain $\hat{\mathcal{U}}^{(1)}_{32} = \hat{\mathcal{V}}^{(1)}_{31}$. We set

$$\sqrt{\omega} = \hat{\mathcal{U}}^{(1)}_{32}, \qquad a = \hat{\mathcal{U}}^{(1)}_{21}\omega, \qquad b = \hat{\mathcal{V}}^{(1)}_{12}\omega.$$

Since $\mathcal{U}$ and $\mathcal{V}$ are $\mathfrak{g}[\lambda]$ valued function on $\mathbb{D}$, we obtain

$$\hat{\mathcal{U}}^{(0)} = \alpha_0\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad \hat{\mathcal{U}}^{(1)} = \begin{pmatrix}0 & 0 & 0\\ \dfrac{a}{\omega} & 0 & 0\\ 0 & \sqrt{\omega} & 0\end{pmatrix},$$

$$\hat{\mathcal{V}}^{(0)} = \beta_0\begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad \hat{\mathcal{V}}^{(1)} = \begin{pmatrix}0 & \dfrac{b}{\omega} & 0\\ 0 & 0 & 0\\ \sqrt{\omega} & 0 & 0\end{pmatrix}.$$

Finally, the compatibility condition

$$[\mathcal{U}(\lambda), \mathcal{V}(\lambda)] - \mathcal{U}(\lambda)_v + \mathcal{V}(\lambda)_u = 0$$

is equivalent to the systems

$$[\hat{\mathcal{U}}^{(1)}, \hat{\mathcal{V}}^{(0)}] - \hat{\mathcal{U}}^{(1)}_v = 0, \qquad [\hat{\mathcal{U}}^{(0)}, \hat{\mathcal{V}}^{(1)}] - \hat{\mathcal{V}}^{(1)}_u = 0,$$
$$[\hat{\mathcal{U}}^{(0)}, \hat{\mathcal{V}}^{(0)}] + [\hat{\mathcal{U}}^{(1)}, \hat{\mathcal{V}}^{(1)}] - \hat{\mathcal{U}}^{(0)}_v + \hat{\mathcal{V}}^{(0)}_u = 0,$$

which yield following equations:

$$\alpha_0 = \frac{\omega_u}{2\omega}, \qquad \beta_0 = \frac{\omega_v}{2\omega}, \qquad a_v = 0, \qquad b_u = 0, \qquad (\log\omega)_{uv} + ab\omega^{-2} = 0.$$

Therefore, ψ describes a modified frame of indefinite improper affine spheres. □

A natural integrable discretization of the system is obtained as follows. The discrete Gauss equations admit a one parameter family of solutions, since the transformation $A \mapsto \mu A$ and $B \mapsto \mu^{-1}B$ with an arbitrary nonzero constant $\mu \in \mathbb{R}$ leave the discrete Gauss–Codazzi equations unchanged. Thus we obtain moving frame equations, where the coefficient matrices also depend on the additional nonzero real parameter μ. We set

$$\Phi(n,m,\lambda)=(F_1-F,F_2-F,\Xi)\begin{pmatrix}\dfrac{1}{\lambda} & 0 & 0\\ 0 & \dfrac{\lambda}{\Omega} & 0\\ 0 & 0 & 1\end{pmatrix},\qquad \lambda=\sqrt[3]{\mu}$$

and assume $\Omega=\det(F_1-F,F_2-F,\Xi)$, so that we obtain a map $\Phi:\mathbb{Z}^2\to G[\lambda]$, which satisfies

$$\Phi(\lambda)^{-1}\Phi(\lambda)_1=\begin{pmatrix}\dfrac{\Omega_1}{\Omega} & 0 & 0\\ \lambda A_1 & \dfrac{\Omega}{\Omega_1} & 0\\ \lambda^2 A_1 & \dfrac{\lambda\Omega}{\Omega_1} & 1\end{pmatrix},$$

$$\Phi(\lambda)^{-1}\Phi(\lambda)_2=\begin{pmatrix}1 & \dfrac{\lambda^{-1}B_2}{\Omega\Omega_2} & 0\\ 0 & 1 & 0\\ \lambda^{-1}\Omega & \dfrac{\lambda^{-2}B_2}{\Omega_2} & 1\end{pmatrix}.$$

Then we obtain the following theorem which gives a discrete analogue of Theorem 6.1.

Theorem 6.2. *Let U and $V:\mathbb{Z}^2\to G[\lambda]$ be two maps of the form*

$$U(\lambda)=\sum_{k=0}^{2}\lambda^k U^{(k)},\qquad V(\lambda)=\sum_{k=0}^{2}\lambda^{-k}V^{(k)}$$

with their coefficients satisfying $U_{21}^{(1)}U_{32}^{(1)}V_{12}^{(1)}V_{31}^{(1)}\neq 0$. Assume that the inverse matrix $U(\lambda)^{-1}$ contains only λ^0 and λ^1 and that $V(\lambda)^{-1}$ contains only λ^0 and λ^{-1}. Assume moreover that the equation

$$U(\lambda)V(\lambda)_1-V(\lambda)U(\lambda)_2=0$$

holds for all loop parameters $\lambda\in S^1$. Let $\Phi:\mathbb{Z}^2\to G[\lambda]$ be a map which satisfies

$$\Phi(\lambda)^{-1}\Phi(\lambda)_1=U(\lambda),\qquad \Phi(\lambda)^{-1}\Phi(\lambda)_2=V(\lambda).$$

Then, there exists uniquely $C\in G^0$ such that the map $\Phi C:\mathbb{Z}^2\to G[\lambda]$ describes a modified frame of discrete indefinite improper affine spheres.

Proof. We define $G^0 \ni C = \mathrm{diag}[\alpha, 1/\alpha, 1]$, where $\alpha = \sqrt{U_{31}^{(2)}/U_{21}^{(1)}}$, and

$$\Omega = \frac{U_{31}^{(2)}V_{32}^{(2)}}{U_{21}^{(1)}V_{12}^{(1)}}, \qquad A_1 = \frac{U_{31}^{(2)}(U_{31}^{(2)})_1}{(U_{21}^{(1)})_1}, \qquad B_2 = \frac{V_{32}^{(2)}(V_{32}^{(2)})_2}{(V_{12}^{(1)})_2},$$

which yield that the map ΦC gives a modified frame of discrete indefinite improper affine spheres. □

Remark 6.3. In general, quadratic elements $U(\lambda) = \sum_{k=0}^{2} \lambda^k U^{(k)}$, $V(\lambda) = \sum_{k=0}^{2} \lambda^{-k} V^{(k)} \in G[\lambda]$ are of the form

$$U(\lambda) = \begin{pmatrix} U_{11}^{(0)} & 0 & 0 \\ \lambda U_{21}^{(1)} & \dfrac{1}{U_{11}^{(0)}} & 0 \\ \lambda^2 U_{31}^{(2)} & \lambda U_{32}^{(1)} & 1 \end{pmatrix}, \qquad V(\lambda) = \begin{pmatrix} V_{11}^{(0)} & \lambda^{-1} V_{12}^{(1)} & 0 \\ 0 & \dfrac{1}{V_{11}^{(0)}} & 0 \\ \lambda^{-1} V_{31}^{(1)} & \lambda^{-2} V_{32}^{(2)} & 1 \end{pmatrix},$$

and their inverses are:

$$U(\lambda)^{-1} = \begin{pmatrix} \dfrac{1}{U_{11}^{(0)}} & 0 & 0 \\ -\lambda U_{21}^{(1)} & U_{11}^{(0)} & 0 \\ \lambda^2 \left(U_{21}^{(1)} U_{32}^{(1)} - \dfrac{U_{31}^{(2)}}{U_{11}^{(0)}} \right) & -\lambda U_{32}^{(1)} U_{11}^{(0)} & 1 \end{pmatrix},$$

$$V(\lambda)^{-1} = \begin{pmatrix} \dfrac{1}{V_{11}^{(0)}} & -\lambda^{-1} V_{12}^{(1)} & 0 \\ 0 & V_{11}^{(0)} & 0 \\ -\dfrac{\lambda^{-1} V_{31}^{(1)}}{V_{11}^{(0)}} & \lambda^{-2}(V_{12}^{(1)} V_{31}^{(1)} - V_{11}^{(0)} V_{32}^{(2)}) & 1 \end{pmatrix}.$$

Therefore, the assumption on the degree of inverse matrices $U(\lambda)^{-1}$, $V(\lambda)^{-1}$ in Theorem 6.2 is equivalent to the following equations:

$$\frac{U_{21}^{(1)} U_{32}^{(1)} - U_{31}^{(2)}}{U_{11}^{(0)}} = 0, \qquad V_{12}^{(1)} V_{31}^{(1)} - V_{11}^{(0)} V_{32}^{(2)} = 0.$$

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